Proof of Makarov's dimension Theorem: the Upper bound

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Thm $f: \mathbb{D} \to \Omega$ - conformal, $\partial\Omega$ - Jordan.
Then $|\partial\Omega| < \infty \iff f' \in H^1$ (i.e. $\sup_r \int |f'(r\zeta)||d\zeta| < \infty$).
Moreover, $|f(A)| = \int_A |f'(\zeta)||d\zeta|$, $|E| = 0 \iff |f(E)| = 0$.

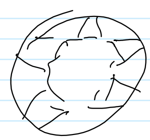
Pf Let $f' \in H^1$. f can be extended to $\overline{\mathbb{D}}$.
$\Lambda_1(\partial\Omega) = \sup_{\text{partitions of } \mathbb{T}} \sum |f(e^{i\theta_j}) - f(e^{i\theta_{j-1}})| = \sup \lim_{r \to 1-} \sum |f(re^{i\theta_j}) - f(re^{i\theta_{j-1}})|$
$\le \lim_{r \to 1-} \int |f'(re^{i\theta})| d\theta = \|f'\|_{H^1}$. Applying it to A, get $|f(A)| \le \lim_{r\to 1-} \int_A |f'(r\zeta)| = \int_A |f'(r\zeta)| d\zeta$.
Conversely, take any partition of $\mathbb{T}$.
$\psi(z) := \sum |f(ze^{i\theta_j}) - f(ze^{i\theta_{j-1}})|$ - subharmonic, so max at some $|z| = 1$.
$(\psi(z) \le |\psi(z)| \le \Lambda_1(\partial\Omega)!$
But $\psi(r)$ is a Riemann sum for $\int |f'(re^{i\theta})| d\theta$, so $\int |f'(re^{i\theta})| d\theta \le \Lambda_1(\partial\Omega)$. Apply to A to get another $\ge$ in 2nd part
So $|E| = 0 \Rightarrow |f(E)| = 0$. But if $|f(E)| = 0 \Rightarrow |f'(\zeta)| = 0$ a.e. $\zeta \in E$.
But $f' \in H^1$! We use Riesz uniqueness Thm to show that $|E| = 0$ ∎

Corollary $\partial\Omega$ - rectifiable $\Rightarrow f'(e^{i\theta})$ exists a.e. as radial or non-tangential limit.

Def $\Gamma_{\alpha,h}(\zeta) = \{ z \in \mathbb{D},\ \text{hdist}(z, [z,\zeta]) < \alpha,\ |z| > h \}$.
f is <u>non-tangentially bounded</u> at ζ if $\exists \alpha, h$: $\sup_{z \in \Gamma_{\alpha,h}(\zeta)} |f(z)| < \infty$.

Thm. Let f be meromorphic in $\mathbb{D}$, for some $E \subset \mathbb{T}$, f is non-tangentially bounded $\forall \zeta \in E$. Then $\forall \zeta \in E$: $\exists \alpha, h$ (depending on ζ!) such that $\exists \lim_{z \in \Gamma_{\alpha,h}(\zeta),\ z \to \zeta} f(z)$.

Pf. $\exists$ K - compact; $|E \setminus K| < \varepsilon$, $K \subset E$, with $\exists \alpha, h, M$ $\forall \zeta \in K$
$\sup_{z \in \Gamma_{\alpha,h}(\zeta)} |f(z)| \le M$.



Take $\Omega = \bigcup_{\zeta \in K} \Gamma_{\alpha,h}(\zeta)$
Then $\|f\|_{\infty,\Omega} \le M$. Let Ω_j - any connected component. $\partial\Omega_j$ - rectifiable, so if $\varphi_j: \mathbb{D} \to \Omega$ - uniformization, then $f \circ \varphi_j \in H^\infty \Rightarrow f \circ \varphi_j$ has non-tangential limits a.e. so $\forall j$ $|\{\zeta \in \mathbb{T} : \zeta \in \partial\Omega_j,\ \lim_{r \to 1-} f(r\zeta) \text{ does not exist}\}| = 0$ ∎

Thm (Privalov). Let $f(z)$ meromorphic in $\mathbb{D}$, $\lim_{z \to \zeta,\ \text{non-tangential}} f(z) = 0$ on a set of positive measure. Then $f \equiv 0$.

Pf Let $E = \{\zeta : \lim_{z \to \zeta,\ \angle} f(z) = 0\}$. Take $\varepsilon > 0$.
Take $K \subset E$, $|K| > 0$; s.t. $\exists \alpha, h$: $|f(\zeta)| < \varepsilon$ on $\Gamma_{\alpha,h}(\zeta)$, $\zeta \in K$.
Define Ω_j - connected component of $\bigcup_{\zeta \in K} \Gamma_{\alpha,h}(\zeta)$. $\exists j : |\partial\Omega_j \cap K| > 0$.
$\exists$ $\varphi_j : \mathbb{D} \to \Omega_j$, $f \circ \varphi_j \in H^\infty$, on $\varphi_j^{-1}(K)$, we have $f \circ \varphi_j \equiv 0$. So
since $|K \cap \partial\Omega_j| = \int_{\varphi^{-1}(K)} |\varphi_j'|$, we have $|\varphi^{-1}(K)| > 0$. So $f \equiv 0$ on $\Omega_j \Rightarrow$
$f \equiv 0$ ∎

Def $\zeta \in \partial\mathbb{D}$ is a Plessner pt for f if $\forall \alpha, h$ $f(\Gamma_{\alpha,h}(\zeta))$ is dense in $\hat{\mathbb{C}}$.

Thm (Plessner). $f \not\equiv$ const, meromorphic. Then $\mathbb{T} = N \sqcup G \cup P$: $|N| = 0$, $\forall \zeta \in P$ – Plessner pt, and $\forall \zeta \in G$ $\exists \Gamma_{\alpha,h}(\zeta)$ with $\lim\limits_{\substack{z \in \Gamma_{\alpha,h}(\zeta) \\ z \to \zeta}} f(z)$ $\exists$ and $\neq \infty$.

Pf. Take P – set of Plessner pts. $E := \mathbb{T} \setminus P$.

Take $\{w_n\}$ – dense in $\mathbb{C}$, $\varepsilon_k \to 0$.

$E_{n,k} := \{\zeta : \exists \Gamma_{\alpha,h}(\zeta) : \forall z \in \Gamma_{\alpha,h}(\zeta) : |f(z) - w_n| > \varepsilon_k\}$.

$E = \bigcup\limits_{n,k} E_{n,k}$ ($\forall \zeta \in E$ is not Plessner).

Consider $\frac{1}{f(z) - w_n}$ – non-tangentially bounded on $E_{n,k}$ (by $\frac{1}{\varepsilon_k}$) $\Rightarrow$ a.e. $\zeta \in E_{n,k}$ $\exists \lim\limits_{z \in \Gamma_{\alpha,h}(\zeta) \to \zeta} f(z)$. $\neq \infty$ a.e. by uniqueness + Lusin

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Vitali Covering Lemma. $E \subset \mathbb{R}^n$, $E \subset \bigcup B(x_\alpha, r_\alpha)$; $\forall \varepsilon > 0$, $x \in E \Rightarrow \exists B(x_\alpha, r_\alpha) \ni x$, $r_\alpha < \varepsilon$. Then one can find disjoint subfamily of balls $B'(x_j, r_j)$: $m_n(E \setminus \bigsqcup B(x_j, r_j)) = 0$.

Pf. WLOG E – bounded (otherwise, consider exhaustion by bounded). Choose B' by induction with $\frac{1}{2}$ largest radius not intersecting previously chosen. Then $\sum m_n(B_k') < \infty$, and if $D_k' := 5B_k'$, then $E \setminus \bigcup\limits_{k=1}^{\ell} B_k \subset \bigcup\limits_{k=\ell+1}^{\infty} D_k'$, so $m(E \setminus \bigcup\limits_{k=1}^{\ell} B_k') \le 5^n \sum\limits_{k=\ell}^{\infty} m(B_k') \to 0$ ∎

Lemma (Milloux) $\exists c > 0$: $\forall$ connected $E \subset \{\frac{1}{2} \le |z| \le 1\}$, $E \cap \{|z| = \frac{1}{2}\} \neq \emptyset$, $E \cap \{|z| = 1\} \neq \emptyset \Rightarrow \omega(E, \mathbb{D} \setminus E, 0) \ge c$.

Remark. Beurling projection Thm $\Rightarrow$ minimal is for $E = \{\frac{1}{2} \le t \le 1\}$ – interval.

Pf. WLOG $\frac{1}{2} \in E$. $\overline{E}$ – complex conjugate of E.

By max. principle,

$$\omega(E, \mathbb{D} \setminus E, z) + \omega(\overline{E}, \mathbb{D} \setminus \overline{E}, z) \ge \omega(E \cup \overline{E}, \mathbb{D} \setminus (E \cup \overline{E}), z)$$

$$\ge \omega([\tfrac{1}{2}, 1], \mathbb{D} \setminus [\tfrac{1}{2}, 1], z).$$ Take $z = 0$, so

$\omega(E, \mathbb{D} \setminus E, 0) = \omega(\overline{E}, \mathbb{D} \setminus \overline{E}, 0)$ ∎

Def. $E \subset \partial\Omega$, $\{z_n\} \subset \Omega$. $\{z_n\}$ are said to be non-tangentially dense on E if $\forall \alpha$: $\forall \zeta \in E$: $\exists z_{n_k} \to \zeta$, $z_{n_k} \in \Gamma_\alpha(\zeta)$.

Lemma (Rohde) Let a sequence $\{z_k\}$ be non-tangentially dense for $E \subset \mathbb{T}$, $f: \mathbb{D} \to \Omega$ – conformal, $w_k := f(z_k)$, $r_k := \operatorname{dist}(w_k, \partial\Omega)$, $B_k := B(w_k, 2r_k)$. $V := \partial\Omega \cap \bigcup B_k$. Then $|E \setminus f^{-1}(V)| = 0$.

Pf. Assume false. W_k := component of w_k of $B_k \cap \Omega$.

$\omega(V, \Omega, w_k) \ge \omega(\partial\Omega, W_k, w_k) \ge c$, by Milloux Lemma.

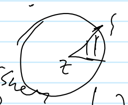
$u(z) := \omega(V, \Omega, f(z))$ – harmonic, $\neq 0$ in $\mathbb{D}$. Has non-tangential limits a.e. on $\mathbb{T}$ (bounded!) $\Rightarrow$ at a.e. $\zeta \in E$ u has non-tangential limit.

Select $z_{n_k} \to \zeta$ non-tangentially, $u(z_{n_k}) \ge c \Rightarrow u(\zeta) \ge c \Rightarrow$ a.e. $\zeta \in E$, $u(\zeta)$ exists as non-tangential limit and $\ge c$.

But $u(z) = \int \chi_{f^{-1}(V)}\, d\omega$, so $u(\zeta) = \chi_{f^{-1}(V)}(\zeta)$ a.e. Thus $|E \setminus f^{-1}(V)| = 0$ ∎

Makarov's Thm - upper bound. Let $\lim_{t\to 0} \frac{h(t)}{t} = 0$, ω – harmonic measure of s.c. domain. Then $\omega \perp \Lambda_h$, i.e. $\exists E$: $\omega(E) = 1$, $\Lambda_h(E) = 0$.
In particular, if $\alpha > 1$, $\dim \omega < \alpha$. $\left(\frac{t^\alpha}{t} \to 0\right)$.

Pf. Let $I(z) := \{\zeta \in \mathbb{T} : z \in \Gamma_{\pi/4}(\zeta)\}$



$f : \mathbb{D} \to \Omega$ - conformal $\Rightarrow$ f' - meromorphic $\overset{\text{Plessner}}{\Longrightarrow}$ $\lim_{z \in \Gamma_{\pi/4}(\zeta) \to \zeta} |f'(z)| < \infty$ a.e. $\zeta \in \mathbb{T}$

Let $E_n := \{\zeta : \lim_{z \in \Gamma_{\pi/4}(\zeta) \to \zeta} |f'(z)| \le n\}$ $E = \cup E_n$.

We will see: $M_h(f(E_n)) = 0$. But $\omega(\cup f(E_n)) = \omega(f(E)) = 1$.

More precisely:

Fix $\varepsilon > 0$. Choose $\delta_n < \frac{\varepsilon}{n}$: for $t < 4n\delta_n$, $\frac{h(t)}{t} < \frac{\varepsilon}{n 2^{n+2}}$.

For $\zeta \in E_n$ $\exists z$: $|I(z)| < \varepsilon$, $|f'(z)| < n$, $\zeta \in I(z)$, and $(1-|z|^2) < \delta_n$.

E_n is covered by such $I(z)$. By Vitali, $\exists z_{n,i}$: $I_{n,i} := I(z_{n,i})$ – disjoint, cover a.e. E_n. $\sum_i |I_{n,i}| \le 2\pi$.

Then $(z_{n,j})_{n,j}$ – non-tangentially dense in $\bigcup_n \left(\bigcup_j I_{n,j}\right)$, and $|E \setminus \bigcup_n \left(\bigcup_j I_{n,j}\right)| = 0$.

Let $w_{n,j} := f(z_{n,j})$, $B_{n,j}$ – as in Rohde's Lemma, $r_{n,j}$ – its radius.

Set $V_{\varepsilon,n} := \partial\Omega \cap \left(\bigcup_j B_{n,j}\right)$. Then, by Rohde's Lemma $|E \setminus f^{-1}\left(\bigcup_n V_{\varepsilon,n}\right)| = 0$.

By Köbe distortion, $2\varepsilon > 2|f'(z_{n,j})|\left(1-|z_{n,j}|^2\right) \ge \frac{r_{n,j}}{4}$. So

$M_h(V_{\varepsilon,n}) \le \sum h(r_{n,j}) \le \sum h\left(2|f'(z_{n,j})|(1-|z_{n,j}|)^2\right) \le \sum \frac{\varepsilon}{n 2^{n+2}} 2|f'(z_{n,j})|(1-|z_{n,j}|) \le$

$\sum \frac{\varepsilon}{n 2^{n+1}} n |I(z_{n,j})| \le \frac{\varepsilon}{2^{n+1}} \sum |I(z_{n,j})| \le \frac{\pi\varepsilon}{2^n}$. So for $V_\varepsilon := \cup V_{\varepsilon,n}$, $M_h(V_\varepsilon) \le \sum M_h(V_{\varepsilon,n}) \le \sum \frac{\pi\varepsilon}{2^n} = \pi\varepsilon$.

Take $V = \cap V_{1/m}$ $\Rightarrow$ $\omega(V) = 1$, but $M_h(V) = 0$ ∎

Can be strengthened:

Thm (Pommerenke) $\exists E \subset \partial\Omega$: $\omega(E) = 1$, E – σ-finite length.

For the proof, need definition:

Def. $w \in \partial\Omega$ is called cone point if $\exists T = T(w)$ – open isosceles triangle in Ω, $(T \subset \Omega)$, with w as the vertex with equal sides.

$K := K_\Omega = \{w \in \partial\Omega : w \text{ is a cone point}\}$.

Lemma. K has σ-finite length.

Pf. Let L_n - countable set of all lines with rational slopes containing a rational point.

$K_n := \{w \in K_n : \exists T_n(w)$ with vertex angle $\frac{\pi}{n}$, has base on $L_n \cap \{|z| < n\}$, $T_n \subset \Omega$, height of $T_n \ge \frac{1}{n}$ and $\le n\}$.

$K = \cup K_n$, K_n - compact $\Rightarrow$ K – F_σ set.

Observe: $\operatorname{dist}(K_n, L_n) \ge \frac{1}{n}$. Let $\Omega_n = \bigcup_{w \in K_n} T_n(w)$.

$\Omega_n \subset \Omega$, has finitely many components (area of $T_n \ge C_n$) $\Omega_{n,j}$.

$\Gamma_{n,j} := \partial \Omega_{n,j}$ - Jordan curves, disjoint (except for end points),
$K_n \subset \bigcup (\Gamma_{n,j} \setminus L_n)$, and $\Gamma_{n,j} \setminus L_n$ is a Lipschitz graph, so it has finite length! ∎

Remark. Actually, on K ω and length are absolutely continuous.

Pf of Pommerenke.

Let $G := \{ \zeta \in \mathbb{T} : \exists f'(\zeta) = \lim_{r \to 1-} f'(r\zeta), f'(\zeta) \neq 0 \}$.

Easy to see: $f(G) \subset K$.

$B := \{ \zeta \in \mathbb{T} : \exists f(\zeta) = \lim_{r \to 1-} f(r\zeta)$ but $\lim_{z \to \zeta \text{ non-tangentially}} |f'(z)| = 0 \}$.

But Plessner + Privalov:

$|G \cup B| = 1$. (A.e. $\zeta \in \mathbb{T} \Rightarrow \zeta \in G \cup B$).

Lemma (Pommerenke) $\exists S \subset f(B)$: 1) $\Lambda_1(S) = 0$ 2) $\omega(S \cup K) = 1$.

Pf. As in pf of Makarov's Thm, find $\{z_{n,j}\}$ such that:

1) $|f'(z_{n,j})| < 2^{-n-3}$

2) $|B \setminus \bigcup I(z_{n,j})| = 0$

3) $\sum_j |I(z_{n,j})| \leq 2\pi$

Take $w_{n,j} := f(z_{n,j})$, $r_{n,j} := \operatorname{dist}(w_{n,j}, \partial\Omega)$.

$B_{n,j} := B(w_{n,j}, 2r_{n,j})$ $\quad V_n := \partial\Omega \cap (\bigcup_j B_{n,j})$

Then $\sum 2r_{n,j} \leq \sum 2|f'(z_{n,j})| (1 - |z_{n,j}|^2) \leq C\pi 2^{-n}$.

So if $V := \bigcap_N \bigcup_{n \geq N} V_n$, then $\Lambda_1(V) = 0$.

Let $S := V \cap f(B)$, then $\Lambda_1(S) = 0$.

By Milloux, $\omega(w_{n,j}, V_n, \Omega) \geq c$ and $\bigcup_{n \geq N} (z_{n,j})$ is non-tangentially dense on B.

So, by Rohde, $|B \setminus \bigcup_{n \geq N} f^{-1}(A \cap V_n)| = 0$ $\forall N$. So

$|B \setminus f^{-1}(S)| = 0 \Rightarrow \omega(f(B) \setminus S) = 0$. So $\omega(S \cup K) = \omega(f(B \cup G)) = 1$ ∎